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603 984

ON THE EVALUATION OF NOISE SAMPLES

Arnold J. F. Siegert \*

P-236

September 7, 1951

17 + \$1.00 to  
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ON THE EVALUATION OF NOISE SAMPLES

Arnold J. P. Siegert

Abstract

↓ Unless the primary source of noise is one of those which are theoretically tractable, the statistical properties of the noise have to be inferred from samples. We have developed some criteria to aid in the decision whether a sample can reasonably be assumed to have come from a Gaussian noise with predetermined parameters. ↑

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Introduction: The optimum design of instruments such as aiming devices requires the knowledge of the statistical properties of the noise which will accompany the signals to which the devices react. Unless the primary source of this noise is one of those which are theoretically completely tractable, such as thermal noise, shot effect, etc., the statistical properties of the noise must be obtained empirically from samples. Since the size of the sample is often limited by the cost of obtaining samples, the problem arises to which extent the statistical properties obtained from a given sample can be relied upon. This problem has been given precise formulation in the statistical literature for samples taken from populations consisting of discrete elements. Since the elements of the sample in our case are the observed values of  $y(t)$  in a time interval  $(0, T)$ , they are neither discrete nor statistically independent. The sample estimates are functionals of  $y(t)$  and the only nontrivial functionals for which - even for a Gaussian random function - the probability distribution is at present known or can be obtained are the linear integral forms<sup>1)</sup> (weighted sample means) and the quadratic integral forms<sup>2)</sup> (such as mean square, correlation function and spectrum of the sample)<sup>3)</sup>.

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- 1) N. Wiener, Acta Math. 55, 117, 1930.
  - 2) M. Kac and A.J.F. Siegert, J. Appl. Physics, 18, 383, 1947 and Ann. Math. Stat. 18, 438, 1947
  - 3) For Markoffian random functions the probability distribution of the largest value in the sample can be obtained (A.J.F. Siegert, Phys. Rev. 81, 617, 1951) and for an especially simple Markoffian random function the Laplace transform of the distribution of the integral of the absolute value and certain other non-linear functionals are known (M. Kac, Trans. Am. Math. Soc. 59, 401, 1946; 65, 1, 1949.)
- 

The expectation value and average square of sample estimates can usually

be obtained for Gaussian random functions, and yield some information about the probability distribution through the Bienaymé - Tchebycheff inequality<sup>4)</sup>

4) c.f. Harald Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, pp. 182-183.

Since most of the above results have been calculated for Gaussian random functions only it seemed of interest to develop the basis for a criterion by which one may judge whether at least the first distribution of the sample is sufficiently close to a Gaussian distribution so that the sample could have come from a Gaussian random function. We define the first distribution  $\mathcal{V}(a, T)$  of a sample  $y(t)$  as that fraction of the interval  $(0, T)$  during which  $y(t) > a$ . We have computed the fluctuation  $S^2 = \langle [\mathcal{V}(a, T) - \phi(a)]^2 \rangle_{AV}$  and the integrals  $S$  and  $S_2$  of the fluctuation over all values of  $a$  without weighting and with a weight function  $G(a)$  resp for Gaussian random functions with first probability distribution  $\phi(a)$  and with arbitrarily given correlation function.

The hypothesis that the sample came from a Gaussian distribution with specified parameters can be rejected if the deviation  $|\mathcal{V}(a, T) - \phi(a)|$  at a predetermined<sup>5)</sup> value  $a$  appreciably exceeds the value  $S$ , or if the integral  $\int_{-\infty}^{\infty} [\mathcal{V}(a, T) - \phi(a)]^2 G(a) da$  appreciably exceeds  $S$  or  $S_2$  resp.

5) "Predetermined" means that the value of  $a$  is chosen without knowledge of the sample, and not, for instance, at the point for which  $|\mathcal{V}(a, T) - \phi(a)|$  has its largest value.

More precisely we know from the Bienaymé - Tchebycheff inequality

$$\text{prob} \left\{ |\mathcal{V}(a, T) - \phi(a)| \geq kS \right\} \leq 1/k^2$$

for any predetermined value of  $a$ , and

$$\text{prob} \left\{ \int_{-\infty}^{\infty} (G(a)) [\mathcal{V}(a, T) - \phi(a)]^2 da \geq kS_{(2)} \right\} \leq 1/k$$

There remains, of course, as always, an arbitrariness in choosing a value for the probability at which one wishes to reject the hypothetical distribution, rather than accept an improbable outcome of the experiment.

# 1) Definitions

The fraction of the time  $T$  during which  $y(t) > a$  is denoted by  $\mathcal{V}(a, T)$  and expressed as

$$(1.1) \quad \mathcal{V}(a, T) = \frac{1}{T} \int_0^T \beta(y(t) - a) dt$$

where  $\beta$  is defined by

$$(1.2) \quad \beta(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Not only the problem of finding the probability distribution of  $\mathcal{V}(a, T)$  but even the special cases  $\text{prob} \left\{ \mathcal{V}(a, T) = 0 \right\}$  and  $\text{prob} \left\{ \mathcal{V}(a, T) = 1 \right\}$  seem to be extremely difficult, except for Markoffian random functions, since their solution is closely related to the solution of the first passage time probability problem.

We define the first and second probability functions of the stationary random function  $y(t)$  by

$$(1.3) \quad \text{prob} \left\{ y_1 \leq y(t_1) < y_1 + dy_1 \right\} = W(y_1) dy_1$$

and

$$(1.4) \quad \text{prob} \left\{ \begin{array}{l} y_1 \leq y(t_1) < y_1 + dy_1 \\ y_2 \leq y(t_2) < y_2 + dy_2 \end{array} \right\} = W_2(y_1, y_2; t_2 - t_1) dy_1 dy_2$$

and obtain

$$\langle \varphi(a, T) \rangle_{AV} = \frac{1}{T} \int_0^T \langle \beta(y(t) - a) \rangle_{AV} dt = \phi(a)$$

where

$$(1.5) \quad \phi(a) \equiv \int_a^{\infty} W(y) dy$$

is the probability that  $y(t)$  exceeds the value  $a$ .

For the second moment we get

$$\begin{aligned} \langle \varphi^2(a, T) \rangle_{AV} &\equiv \left\langle \left( \frac{1}{T} \int_0^T \beta(y(t) - a) dt \right)^2 \right\rangle_{AV} \\ (1.6) \quad &= \frac{1}{T^2} \int_0^T \int_0^T dt_1 dt_2 \langle \beta(y(t_1) - a) \beta(y(t_2) - a) \rangle_{AV} \\ &= \frac{1}{T^2} \int_0^T \int_0^T dt_1 dt_2 \int_a^{\infty} \int_a^{\infty} W_2(y_1, y_2; t_2 - t_1) dy_1 dy_2 \\ &= \frac{2}{T} \int_0^T dt \left(1 - \frac{t}{T}\right) \int_a^{\infty} \int_a^{\infty} W_2(y_1, y_2; t) dy_1 dy_2 \end{aligned}$$

## 2. Rough Approximations

For orientation we consider first the oversimplified<sup>6)</sup> case

$$(2.1) \quad W_2(y_1, y_2; t_2 - t_1) = \begin{cases} W(y_1) \delta(y_2 - y_1) & \text{for } |t_2 - t_1| \leq \tau \\ W(y_1) W(y_2) & \text{for } |t_2 - t_1| > \tau \end{cases}$$

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6) This is not a realistic assumption, since it yields for the normalized correlation function

$$\rho(t_2 - t_1) \equiv (\langle y(t_2) y(t_1) \rangle_{AV} - \langle y \rangle_{AV}^2) / (\langle y^2 \rangle_{AV} - \langle y \rangle_{AV}^2)$$

the values

$$\rho(t) = \begin{cases} 1 & \text{for } |t| \leq \tau \\ 0 & \text{for } |t| > \tau \end{cases}$$

This is not a possible correlation function, since its Fourier transform can assume negative values.

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A simple geometric consideration yields for this case

$$(2.2) \quad \langle \mathcal{V}^2(a, T) \rangle_{AV} = (1 - \frac{\tau}{T})^2 \phi^2(a) + (1 - (1 - \frac{\tau}{T})^2) \phi(a)$$

so that the fluctuation becomes

$$(2.3) \quad \langle \mathcal{V}^2(a, T) \rangle_{AV} - \langle \mathcal{V}(a, T) \rangle_{AV}^2 = (1 - (1 - \frac{\tau}{T})^2) (\phi(a) - \phi^2(a)) \\ = 2 \frac{\tau}{T} (1 - \frac{\tau}{2T}) (\phi(a) - \phi^2(a))$$

As was to be expected the fluctuation vanishes for  $\frac{\tau}{T} \rightarrow 0$  and becomes equal to  $\phi(a) - \phi^2(a)$  for  $\frac{\tau}{T} \rightarrow 1$ .

A rough idea of the distribution of  $\mathcal{V}(a, T)$  can be obtained by dividing the interval  $T$  into  $m$  equal parts, choosing  $m$  such that the values of  $y(t)$  at the centerpoints  $t_j$  of the intervals can be considered <sup>approximately</sup> as independent random variables. A set of independent random variables  $\mathcal{V}_j$  is then



defined such that  $\mathcal{V}_j = 0$  or 1 if  $\beta(y(t_j) - a) = 0$  or 1, respectively.

The mean  $\mathcal{V}_{(n)} = \frac{1}{n} \sum_{j=1}^n \mathcal{V}_j$  is an approximation for  $\mathcal{V}(a, T)$

The probability distribution for  $\mathcal{V}_{(n)}$  is then

$$(2.4) \quad \text{prob} \left\{ \mathcal{V}_{(n)} = \frac{n}{n} \right\} = \binom{n}{n} \rho^n (1 - \rho)^{n-n}$$

with mean  $\rho(a)$  and fluctuation  $S^2 = n^{-1}(\rho - \rho^2)$ . This is in agreement with Eq. (2.3) for  $\tau/T \ll 1$ .

### 3. Calculation of $\langle \mathcal{V}^2(a, T) \rangle_{AV}$ for Gaussian Random Functions

For the Gaussian random function we have

$$(3.1) \quad W(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}$$

$$(3.2) \quad W_2(y_1, y_2, t) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot e^{-\frac{y_1^2 + y_2^2 - 2\rho y_1 y_2}{2\sigma^2(1-\rho^2)}}$$

To evaluate the integral in Eq. 1.6 we expand  $W_2$  using the equation<sup>7)</sup>

$$(3.3) \quad \sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{n!} \rho^n = \frac{1}{\sqrt{1-\rho^2}} \cdot e^{-\frac{\rho^2(x^2+y^2) - 2\rho xy}{2(1-\rho^2)}}$$

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7) H. Gruber, ref. 4, p. 133

and obtain

$$(3.4) \quad W_2(y_1, y_2, t) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \cdot e^{-\left[ \frac{y_1^2 + y_2^2}{2\sigma^2} + \frac{\rho^2(y_1^2 + y_2^2) - 2\rho y_1 y_2}{2\sigma^2(1-\rho^2)} \right]}$$

$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{y_1^2 + y_2^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{H_n(y_1/\sigma) H_n(y_2/\sigma)}{n!} \rho^n$$

where  $H_n(x)$  are the Hermite polynomials. With

$$(3.5) \quad \int_a^\infty \frac{dy}{\sigma} e^{-\frac{y^2}{2\sigma^2}} H_n\left(\frac{y}{\sigma}\right) = e^{-\frac{a^2}{2\sigma^2}} H_{n-1}(a/\sigma) \quad \text{for } n > 0$$

$$= \sqrt{2\pi} \delta(a) \quad \text{for } n = 0$$

we get

$$(3.6) \quad \iint_a^\infty W_2(y_1, y_2; t) dy_1 dy_2 = \delta^2(a) + \frac{1}{2\pi} e^{-\frac{a^2}{2\sigma^2}} \sum_{n=1}^\infty \frac{H_{n-1}^2(a/\sigma)}{n!} \rho^n$$

$$= \delta^2(a) + \frac{1}{2\pi} e^{-\frac{a^2}{2\sigma^2}} \int_0^\infty \sum_{n=1}^\infty \frac{H_{n-1}^2(a/\sigma)}{(n-1)!} r^{n-1} dr$$

$$= \delta^2(a) + e^{-\frac{a^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^\infty \sum_{n=0}^\infty \frac{H_n^2(a/\sigma)}{n!} r^n dr$$

$$= \delta^2(a) + \frac{1}{2\pi} \int_0^\infty \frac{dr}{\sqrt{1-r^2}} e^{-\frac{a^2}{2\sigma^2} \frac{1}{1+r}}$$

and

$$(3.7) \quad S^2(a, t) \equiv \langle \psi^2(a, T) \rangle_{Av} - \langle \psi(a, T) \rangle_{Av}^2 = \frac{1}{\pi T} \int_0^T \left(1 - \frac{t}{T}\right) dt \int_0^\infty \frac{dr}{\sqrt{1-r^2}} e^{-\frac{a^2}{2\sigma^2} \frac{1}{1+r}} \rho(t)$$

$S^2$  can be written in terms of single integrals in the form

$$(3.8) \quad S^2 = \frac{\rho(T)}{2\pi} \int_0^\infty \frac{dr}{\sqrt{1-r^2}} e^{-\frac{a^2}{2\sigma^2(1+r)}} - \frac{1}{\pi T} \int_0^T t \left(1 - \frac{t}{2T}\right) \frac{d\rho}{dt} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{a^2}{2\sigma^2} \frac{1}{1+\rho}} dt$$

Usually,  $\rho(T)$  will be so small that the first integral can be neglected.

With the oversimplified form of  $\rho(t)$ :

$$(3.9) \quad \rho(t) = \begin{cases} 1 & \text{for } |t| \leq \tau \\ 0 & \text{for } |t| > \tau \end{cases}$$

we get

$$(3.10) \quad S_1^2 \equiv \frac{1}{\pi T} \int_0^\tau \left(1 - \frac{t}{T}\right) dt \int_0^1 \frac{dr}{\sqrt{1-r^2}} \cdot -\frac{a^2}{\sigma^2} \frac{1}{1+r}$$

$$= \frac{2T}{T} \left(1 - \frac{\tau}{2T}\right) \cdot \frac{1}{2\pi} \int_0^1 \frac{dr}{\sqrt{1-r^2}} \cdot -\frac{a^2}{\sigma^2} \frac{1}{1+r}$$

The identity

$$\frac{1}{2\pi} \int_0^1 \frac{dr}{\sqrt{1-r^2}} \cdot -\frac{a^2}{\sigma^2(1+r)} = \phi(a) - \phi^2(a)$$

is easily verified.

A convenient expansion for  $a \ll \sigma$  is obtained by changing to the variable

$$(3.11) \quad y = \sqrt{(1-r)/(1+r)}$$

and writing

$$(3.12) \quad \int_0^\infty \frac{dr}{\sqrt{1-r^2}} \cdot -\frac{a^2}{\sigma^2(1+r)} = 2 \int_{\sqrt{\frac{1-\rho}{1+\rho}}}^1 \frac{dy}{1+y^2} \cdot -\frac{(1+y^2)a^2}{2\sigma^2}$$

we get

$$(3.13) \quad S^2(a, T) \cong \frac{1}{\pi T} e^{-\frac{a^2}{2\sigma^2}} \int_0^T \left[ \left(1 + \frac{a^2}{2\sigma^2}\right) \arcsin \rho - \frac{a^2}{\sigma^2} \left(1 - \sqrt{\frac{1-\rho}{1+\rho}}\right) \right] \left(1 - \frac{t}{T}\right) dt$$

where terms of order  $\left(\frac{a}{\sigma}\right)^4$  have been neglected under the integral.

To judge the sample distribution in its entirety it may also be useful to compute from the data the quantity

$$(3.14) \quad S^* \equiv \int_{-\infty}^{\infty} \left[ \mathcal{W}(a, T) - \phi(a) \right]^2 da$$

and compare it with its expected value  $S_0$ . We have

$$\begin{aligned} (3.15) \quad S &= \langle S^* \rangle_{AV} = \frac{1}{\pi T} \int_0^T \left(1 - \frac{t}{T}\right) dt \int_0^{\rho(t)} \frac{dr}{\sqrt{1-r^2}} \sqrt{\pi \sigma^2 (1+r)} \\ &= \frac{\sigma}{T\sqrt{\pi}} \int_0^T \left(1 - \frac{t}{T}\right) dt \int_0^{\rho(t)} \frac{dr}{\sqrt{1-r}} \\ &= \frac{2\sigma}{T\sqrt{\pi}} \int_0^T \left(1 - \frac{t}{T}\right) dt \left(1 - \sqrt{1-\rho(t)}\right) \end{aligned}$$

For  $\rho(t) = e^{-\beta|t|}$  (Markoffian Gaussian Random Function) the integral has been evaluated in App. I, neglecting terms smaller than  $\frac{2\sigma}{\beta T \sqrt{\pi}} e^{-\beta T}$ .

The result is

$$(3.16) \quad S \cong \frac{2\sigma}{\beta T \sqrt{\pi}} (0.61 - 0.54 (\beta T)^{-1})$$

in this approximation.

Another test which places more emphasis on the deviations of the sample distribution from the Gaussian at large values of  $\left| \frac{a}{\sigma} \right|$  is obtained by computing from the data the quantity <sup>8)</sup>

$$(3.17) \quad S_2^* = \int_{-\infty}^{\infty} \left[ \mathcal{O}(a, T) - \phi(a) \right]^2 \frac{\frac{a^2}{2\sigma^2}}{\sqrt{2\pi\sigma^2}} da$$

Denoting by  $S_2$  the expected value of this quantity we have

$$\begin{aligned} (3.18) \quad S_2 &= \frac{1}{\pi T} \int_0^T \left(1 - \frac{t}{T}\right) dt \int_0^{\rho} \frac{dr}{\sqrt{1-r^2}} \int_{-\infty}^{\infty} \frac{da}{\sqrt{2\pi\sigma^2}} \cdot \frac{-\frac{a^2}{2\sigma^2} \left(\frac{2}{1+r} - r\right)}{1} \\ &= \frac{1}{\pi T} \int_0^T \left(1 - \frac{t}{T}\right) dt \int_0^{\rho} \frac{dr}{\sqrt{1-r^2}} \sqrt{\frac{1+r}{1-r}} \\ &= -\frac{1}{\pi T} \int_0^T \left(1 - \frac{t}{T}\right) dt \lg [1 - \rho(t)] \end{aligned}$$

For the Markoffian case we have (neglecting terms of order  $e^{-\beta T}$ ) the result

$$(3.19) \quad S_2' = \pi^{-1} (\beta T)^{-1} \left[ \zeta(2) - (\beta T)^{-1} \zeta(3) \right]$$

(see Appendix III) with  $\zeta(2) = \frac{\pi^2}{6} = 1.65$  and  $\zeta(3) = 1.20$

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8) Approximative formulas for the contributions to this integral from the regions in which  $\mathcal{O}(a, T) = 0$  or 1 are given in Appendix II.

#### 4. Discussion

The results of our calculations state in various forms how far the empirical first distributions of samples taken from a Gaussian noise with specified parameters will deviate from the true first distribution of this Gaussian noise. They can thus be used to judge whether a sample of unknown source can reasonably be assumed to have come from a Gaussian noise with predetermined parameters. Our results can not in all cases be used to judge whether the sample can reasonably be assumed to have come from a Gaussian noise with parameters obtained from the sample itself. To obtain criteria for the latter decision would require the calculation of quantities such as

$$(4.1) \quad \langle [\varphi(a, T) - \varphi(a, \mu^*, \sigma^*)]^2 \rangle_{AV}$$

where  $\varphi(a, \mu^*, \sigma^*)$  is the Gaussian distribution with mean  $\mu^*$  and standard deviation  $\sigma^*$ , which are both computed from - and thus functionals of - the sample function  $y(t)$ . Expectation values such as (4.1) are much more difficult to calculate than our averages, and may not be obtainable in a useful form.

An answer to the second question above can however still be obtained from our results, if the size  $T$  of the sample is sufficiently large, such that its mean  $\mu^*$  and standard deviation  $\sigma^*$  become reliable estimates of the true mean  $\mu$  and standard deviation  $\sigma$ . In the computation of  $[\varphi(a, T) - \varphi(a)]^2$ ,  $S^*$  and  $S_2^*$  from the sample function, one can then use various combinations of values  $\mu$  and  $\sigma$  deviating from  $\mu^*$  and  $\sigma^*$  resp. by their respective expected errors (which are not difficult to estimate). If the various values of  $[\varphi(a, T) - \varphi(a)]^2$ ,  $S^*$  and  $S_2^*$  thus computed all lead to the same conclusion, our results apply to the second question. We believe that the question whether a sample can have come from a Gaussian noise can reasonably be asked only in this case, when at least the estimates of the mean and standard deviation obtained from the sample are sufficiently reliable.

# APPENDIX I

To evaluate the integral

$$\int_0^T (1 - \frac{t}{T}) dt (1 - \sqrt{1 - e^{-\beta t}}) dt$$

we note that we neglect terms smaller than  $\beta^{-1} e^{-\beta T}$  if we extend the integration to infinity, since

$$1 - \sqrt{1 - e^{-\beta T}} \leq e^{-\beta T}$$

for real, non-negative  $t$  and  $\beta$ . We then have

$$\begin{aligned} I &= \int_0^\infty (1 - \frac{t}{T})(1 - \sqrt{1 - e^{-\beta t}}) dt = \left\{ (1 + \frac{1}{T} \frac{\partial}{\partial \lambda}) \int_0^\infty e^{-\lambda t} (1 - \sqrt{1 - e^{-\beta t}}) dt \right\}_{\lambda=0} \\ &= \left\{ (1 + \frac{1}{T} \frac{\partial}{\partial \lambda}) \left( \frac{1}{\lambda} - \frac{1}{\beta} \int_0^1 x^{\frac{\lambda}{\beta}-1} \sqrt{1-x} dx \right) \right\}_{\lambda=0} \\ &= \left\{ (1 + \frac{1}{T} \frac{\partial}{\partial \lambda}) \left( \frac{1}{\lambda} - \frac{1}{\beta} \frac{\Gamma(\frac{\lambda}{\beta}) \Gamma(\frac{3}{2})}{\Gamma(\frac{\lambda}{\beta} + \frac{3}{2})} \right) \right\}_{\lambda=0} \\ &= \left\{ \beta^{-1} \left[ 1 + (\beta T)^{-1} \frac{\partial}{\partial u} \right] \left( u^{-1} - \frac{\Gamma(u) \Gamma(3/2)}{(u + \frac{1}{2}) \Gamma(u + \frac{1}{2})} \right) \right\}_{u=0} \end{aligned}$$

From

$$\frac{d}{du} \lg \frac{\Gamma(u+1/2)}{\Gamma(u)} = 2 \sum_{k=0}^{\infty} \frac{(-)^k}{2u+k}$$

one obtains<sup>+</sup>

<sup>+</sup>) Detailed derivation in RAND Corporation Report RM-447, "On the Roots of Markoffian Random Functions", A.J.F. Siegert.

$$\frac{\Gamma(s)}{\Gamma(s+1/2)} = \pi^{-1/2} s^{-1} \exp \left\{ - \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} (is)^n \right\}$$

with  $\kappa_1 = 2i \lg 2$

$$\text{and } \kappa_n = (n-1)! (2i)^n (1-2^{1-n}) \zeta(n) \quad \text{for } n \geq 2 \text{ where}$$

$\zeta(n)$  is the Riemann Zeta function<sup>++</sup>)

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++) See e.g. Jahnke Emde, Tables of Functions p. 269 ff.

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We thus have

$$\frac{\Gamma(s)}{\Gamma(s+1/2)} = \pi^{-1/2} s^{-1} (1 - i \kappa_1 s + s^2 \frac{\kappa_2 - \kappa_1^2}{2} + \dots)$$

and

$$\begin{aligned} s^{-1} - \frac{\Gamma(3/2)\Gamma(s)}{(s+1/2)\Gamma(s+1/2)} &= s^{-1} \left\{ 1 - (1 - 2s + 4s^2 + \dots) \right. \\ &\quad \left. (1 - i \kappa_1 s + s^2 \frac{1}{2} (\kappa_2 - \kappa_1^2) + \dots) \right\} \\ &= (2 + i \kappa_1) - s (4 + \frac{1}{2} (\kappa_2 - \kappa_1^2) + 2i \kappa_1) \end{aligned}$$

and

$$\begin{aligned} I &= \beta^{-1} \left\{ (2 + i \kappa_1) - (\beta T)^{-1} \left[ 4 + \frac{1}{2} (\kappa_2 - \kappa_1^2) + 2i \kappa_1 \right] \right\} \\ &= \beta^{-1} \left\{ 2(1 - \lg 2) - (\beta T)^{-1} \left[ 4(1 - \lg 2) - \frac{\pi^2}{6} + 2(\lg 2)^2 \right] \right\} \\ &= \beta^{-1} \left\{ .61 \quad -.54 \quad (\beta T)^{-1} \right\} \end{aligned}$$



# APPENDIX II

Outside of the range of the sample, defined by  $\mathcal{V}(a, T) = 1$  for all  $a \leq l$  and  $\mathcal{V}(a, T) = 0$  for all  $a \geq u$  the integral (3.17) can be obtained analytically as a series in powers of  $(-\sigma/l)$  and  $(\sigma/u)$ . We have

$$\begin{aligned} S^* \equiv & \int_l^u \left[ \mathcal{V}(a, T) - \phi(a) \right]^2 \frac{a^2/2\sigma^2}{\sqrt{2\pi\sigma^2}} da + \int_u^\infty \phi^2(a) \frac{a^2/2\sigma^2}{\sqrt{2\pi\sigma^2}} da \\ & + \int_{-\infty}^l \left[ 1 - \phi(a) \right]^2 \frac{a^2/2\sigma^2}{\sqrt{2\pi\sigma^2}} da, \end{aligned}$$

where

$$\begin{aligned} \phi(a) &\equiv \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^\infty e^{-\frac{y^2}{2\sigma^2}} dy = \frac{1}{\sqrt{\pi}} \int_{a/\sigma\sqrt{2}}^\infty e^{-x^2} dx = \frac{1}{2} \left[ 1 - \frac{2}{\sqrt{\pi}} \int_0^{a/\sigma\sqrt{2}} e^{-x^2} dx \right] \\ &\approx \frac{1}{\sqrt{2\pi}} \frac{\sigma}{a} e^{-\frac{a^2}{2\sigma^2}} \end{aligned}$$

valid for  $a \gg \sigma$ , and

$$I_u \equiv \int_u^\infty \phi^2(a) \frac{a^2/2\sigma^2}{\sqrt{2\pi\sigma^2}} da \approx \sigma (2\pi)^{-3/2} \int_u^\infty \frac{da}{a^2} e^{-\frac{a^2}{2\sigma^2}}$$

which by integration by parts and use of the asymptotic series for the error function yields

$$\begin{aligned} I_u &\approx (2\pi)^{-3/2} (\sigma/u)^3 e^{-\frac{u^2}{2\sigma^2}} \\ &\approx \frac{1}{2\pi} \left( \frac{\sigma}{u} \right)^2 \phi(u) \end{aligned}$$

valid for  $u \gg \sigma$  and

$$I_l \equiv \int_{-\infty}^l \left[ 1 - \phi(a) \right]^2 \frac{a^2/2\sigma^2}{\sqrt{2\pi\sigma^2}} da = \int_{-\infty}^l \phi^2(a) \frac{a^2/2\sigma^2}{\sqrt{2\pi\sigma^2}} da \approx \frac{1}{2\pi} \left( \frac{\sigma}{l} \right)^2 \phi(-l)$$

valid for  $-l \gg \sigma$ .

# APPENDIX III

To evaluate the integral

$$\int_0^T (1 - t/T) dt \lg(1 - e^{-\beta t})$$

we note that the integral can be extended to  $\infty$  incurring an error of order  $e^{-\beta T}$  only.

We then need - for  $S = 3$  and  $S = 2$  - the integral

$$\begin{aligned} \int_0^\infty t^{s-2} dt \lg(1 - e^{-\beta t}) &= \left[ \frac{t^{s-1}}{s-1} \lg(1 - e^{-\beta t}) \right]_0^\infty \\ &= -\frac{1}{s-1} \int_0^\infty t^{s-1} \beta dt e^{-\beta t} (1 - e^{-\beta t})^{-1} \\ &= -\frac{\beta^{1-s}}{s-1} \int_0^\infty x^{s-1} dx e^{-x} (1 - e^{-\beta t})^{-1} \end{aligned}$$

The last integral is equal to  $\Gamma(s) \zeta(s)^{+}$

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+) E. T. Whittaker and G. N. Watson, Modern Analysis, Cambridge, University Press, 1935, p. 266

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so that

$$\int_0^\infty (1 - \frac{t}{T}) dt \lg(1 - e^{-\beta t}) = -\beta^{-1} \zeta(2) + \beta^{-2} T^{-1} \zeta(3)$$